# DIFFUSION AS A DOMINANT DISSIPATIVE MECHANISM IN SUPERCONDUCTING NEUTRON STARS

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## ABSTRACT

A neutron star (NS) can start oscillating as a result of an internal instability or external perturbation. By confronting various observational manifestations of such oscillations with the theoretical models one can constrain the properties of the superdense stellar matter. Whether an oscillation mode can actually be excited or not depends on the interplay between the excitation rate and on how efficiently dissipative mechanisms in the stellar matter counteract this excitation. Generally, at temperatures $T \lesssim 5\times10^8$ K, it is the shear viscosity, that appears to be the strongest one, and other mechanisms (e.g., thermal conductivity and bulk viscosity) can be ignored. At the same time, although the chemical composition of the stellar matter is rather complex and includes different particle species (neutrons, electrons, protons, muons, possibly hyperons and quarks), the effect of particle diffusion on NS oscillations, as far as we know, has not yet been investigated. Here we take a first look into the role of the diffusion as a dissipative agent that can damp NS oscillations. As we demonstrate, in superconducting matter diffusion becomes the leading dissipative mechanism that strongly accelerates the dissipation of various oscillation modes and thus makes questionable our current vision of a number of aspects of NS life.

## Oscillation damping times

Let us consider a small periodic perturbation with frequency $\omega$, propagating in the equilibrium stellar matter. Let $E$ be the mechanical energy of the perturbation, and $\dot{E}$ be the averaged over the period of the oscillation energy loss rate, associated with any dissipative mechanism under consideration. One of the ways to estimate the effect of this mechanism on the dynamics of the perturbation is to calculate the corresponding damping time [1]

$$\tau = -\frac{2E}{\dot{E}}. \tag{1}$$

Below in order to estimate the role of diffusion as a dissipative mechanism we compare the damping time $\tau_{\rm diff}$ due to diffusion with the damping time $\tau_\eta$ due to shear viscosity for a number of different neutron star oscillation modes. To compute the energy loss rate due to shear viscosity $\dot{E}_\eta$ we employ the explicit formulas from Landau & Lifshitz [1]. The calculation of the energy loss rate due to diffusion $\dot{E}_{\rm diff}$ will be discussed below.

## Stellar matter

Here we consider the unmagnetized degenerate stellar matter, consisting of neutrons ($n$), protons ($p$), electrons ($e$) and muons ($\mu$). Neutrons and protons in neutron star interiors can be in superfluid/superconducting state [2]. For simplicity we treat neutrons in our study as normal, since, according to microscopic calculations [3], the maximum critical temperature $T_{cn}$ of neutron superfluidity onset is substantially lower than the proton one, $T_{cp}$. For protons we consider two possibilities of either being normal (normal matter) or strongly superconducting (superconducting matter, where all protons form Cooper pairs). Each particle species $\alpha = n,\ p,\ e$ or $\mu$ is characterized by the electric charge $e_\alpha$, number density $n_\alpha$ and relativistic chemical potential $\mu_\alpha$. In normal matter they have individual velocities $\boldsymbol{v}_\alpha$, while in superconducting matter protons are divided into Bogoliubov thermal excitations with velocity $\boldsymbol{v}_p$ and density $n_{p\ \rm ex}$, and superconducting protons with velocity $\boldsymbol{v}_{sp}$ and density $(n_p - n_{p\ \rm ex})$. In *strongly* superconducting matter $n_{p\ \rm ex}=0$ and only one proton-associated velocity field $\boldsymbol{v}_{sp}$ is left.

For simplicity we show only the equations for the $npe$-matter, but analogous calculations can be performed for $npe\mu$-matter. In $npe$-matter thanks to the electromagnetic interaction protons and electrons move together, so that, to a very high precision, their number densities and currents coincide [4]:

$$n_e = n_p, \qquad n_e\boldsymbol{v}_e = n_{p\ \rm ex}\boldsymbol{v}_p + (n_p - n_{p\ \rm ex})\boldsymbol{v}_{sp}. \tag{2}$$

An unperturbed matter is supposed to be in $\beta$-equilibrium, $\delta\mu_0 \equiv \mu_{n0} - \mu_{p0} - \mu_{e0} = 0$ (hereafter index "0" refers to the unperturbed quantities). Beta-processes are too slow [5] to affect oscillations in sufficiently cold matter and will be neglected in what follows.

## Oscillation equations in homogeneous matter

Here we discuss the oscillation equations for two scenarios, the first of normal matter and the second of strongly superconducting matter, suitable for not too young NSs. For simplicity we discuss only the equations in the homogeneous matter, but analogous calculations can be performed for the global neutron star oscillation modes.

In the hydrodynamic limit, relevant in neutron stars, the system of equations describing small oscillations in homogeneous degenerate unmagnetized matter consists of the number density conservation laws

$$\frac{\partial n_\alpha}{\partial t} + n_{\alpha0}{\rm div}\ \boldsymbol{v}_\alpha = 0 \tag{3}$$

and multifluid hydrodynamic equations, accounting for particle diffusion. According to [4] the friction force between the species $\alpha$ and $\beta$ can be written as $J_{\alpha\beta}\boldsymbol{w}_{\alpha\beta}$, where $\boldsymbol{w}_{\alpha\beta} \equiv \boldsymbol{v}_\alpha - \boldsymbol{v}_\beta$ and $J_{\alpha\beta}$ are the corresponding relative velocity and momentum transfer rate, respectively. Then the dynamics of oscillations in the homogeneous *normal* matter is governed by the following linearized Euler-like equations [4, 6]:

$$\frac{n_{\alpha0}\mu_{\alpha0}}{c^2}\frac{\partial\boldsymbol{v}_\alpha}{\partial t} = e_\alpha n_{\alpha0}\boldsymbol{E} - n_{\alpha0}\boldsymbol{\nabla}\mu_\alpha - \sum_\beta J_{\alpha\beta}\boldsymbol{w}_{\alpha\beta}, \tag{4}$$

where $\boldsymbol{E}$ is the electric field and $c$ the speed of light. In the case of *strongly superconducting* matter Eq. (4) for protons should be replaced with the Josephson equation for the superconducting proton component [7]

$$\frac{\mu_{p0}}{c^2}\frac{\partial\boldsymbol{v}_{sp}}{\partial t} = e_p\boldsymbol{E} - \boldsymbol{\nabla}\mu_p. \tag{5}$$

The hydrodynamic limit corresponds to large values of $J_{\alpha\beta}$ such that $w_{\alpha\beta}$ is much smaller than a typical hydrodynamic velocity $v$. Thus, in what follows $w_{\alpha\beta}$ can be set to zero *unless* it is multiplied by $J_{\alpha\beta}$.

## The effect of diffusion

The energy leakage due to diffusion equals the work done by the friction force per unit time and can be written as an integral over the stellar volume $V$ [4, 8]

$$\dot{E}_{\rm diff} = -\frac{1}{2}\int\sum_{\alpha\beta}J_{\alpha\beta}w^2_{\alpha\beta}\,{\rm d}V. \tag{6}$$

Relative velocities in this equality depend on the efficiency of particle collisions (more effective collisions correspond to smaller relative velocities) and should be expressed through other variables describing the perturbation.

In the case of *normal* matter we sum up Eq.(4) divided by $n_{\alpha0}$ for protons and electrons and subtracting Eq.(4) for neutrons divided by $n_{n0}$, we, using Eq. (2) , the fact that $w_{\alpha\beta}\approx 0$ (unless multiplied by $J_{\alpha\beta}$) and the $\beta$-equilibrium condition $\delta\mu_0 = 0$, arrive at the equation

$$\boldsymbol{\nabla}\delta\mu = -\left[\frac{J_{np}}{n_{p0}} + \frac{J_{en}}{n_{e0}} + \frac{J_{np}+J_{en}}{n_{n0}}\right]\boldsymbol{w}_{np}. \tag{7}$$

This allows us to express the relative velocities through $\delta\mu$ and transform the energy loss rate due to diffusion in *normal* matter (6) into

$$\dot{E}_{\rm diff} \approx -\int\frac{1}{J_{np}}\left[\frac{n_{n0}n_{e0}}{n_{b0}}\boldsymbol{\nabla}\delta\mu\right]^2{\rm d}V, \tag{8}$$

where $n_b = n_n + n_p$ is the baryon number density. Here we neglected $J_{en}$ compared to $J_{np}$, since in normal matter collisions between protons and neutrons are extremely efficient due to strong interactions and $J_{np}/J_{en}\sim 10^5$ [6].

In superconducting matter only thermal excitations can scatter from other particle species [9]. Since in *strongly superconducting* matter $n_{p\ \rm ex}=0$, scattering processes involving protons are suppressed, so that the proton-related momentum transfer rates tend to zero, $J_{p\alpha}=0$. In these conditions Eq. (6) reduces to

$$\dot{E}_{\rm diff} = -\int\frac{1}{J_{en}}\left[\frac{n_{n0}n_{e0}}{n_{b0}}\boldsymbol{\nabla}\delta\mu\right]^2{\rm d}V. \tag{9}$$

To derive (9) we make use of Eq. (2) with $n_{p\rm ex}=0$, Eq. (4) for neutrons and electrons, Eq. (5), and the equality $\delta\mu_0 = 0$.

When dissipation is weak, the imbalance $\delta\mu$ in Eqs.(8) and (9) can be calculated using the equations of nondissipative hydrodynamics, which are the same in normal and strongly superconducting matter. Then Eqs. (8) and (9) imply that dissipation in strongly superconducting matter is by a factor of $\sim J_{np}/J_{en}\sim 10^5$ more efficient than in normal matter. It can be shown that Eqs. (8) and (9) are equally applicable to the inhomogeneous $npe$-matter of Newtonian stars. Similar equations valid in General Relativity (GR) can be derived within the framework of relativistic multi-fluid dissipative hydrodynamics developed in [6, 10].

## Microphysical input

In all numerical calculations we employ the BSk24 equation of state [11], allowing for muons, and adopt shear viscosity coefficients and momentum transfer rates from [12] and [6]. In the case of superconducting matter we neglect the effect of proton superconductivity on $J_{en}$. We assume that protons in superconducting matter are *strongly* superconducting, while neutrons are normal. To calculate global stellar oscillation modes we consider a three-layer NS consisting of the barotropic crust, $npe$ outer core, and $npe\mu$ inner core. We employ the NS model with the mass $M = 1.4M_\odot$ and redshifted internal stellar temperature $T^\infty = 10^8$ K.

## Results: sound waves

In order to find $\tau_{\rm diff}$ we note, that in degenerate $npe$-matter the chemical imbalance $\delta\mu$ depends only on the number densities $n_n$, $n_p$, and $n_e$. For sound waves in nondissipative medium the hydrodynamic velocity $v = v_0\cos(kx-\omega t)$ and number density perturbations $\delta n_\alpha = \delta n_{\alpha0}\cos(kx-\omega t)$, where $k$ is the wave number, are because of number density conservation laws related as $\delta n_\alpha = n_{\alpha0}kv/\omega$. Therefore, we have

$$\delta\mu = \sum_\alpha\frac{\partial\delta\mu}{\partial n_\alpha}\delta n_\alpha = \sum_\alpha\frac{\partial\delta\mu}{\partial n_\alpha}\frac{n_{\alpha0}vk}{\omega}. \tag{10}$$

Using that for sound waves the oscillation energy per unit volume is

$$E = \sum_\alpha\frac{\mu_{\alpha0}n_{\alpha0}v_0^2}{2c^2}, \tag{11}$$

we find

$$\tau_{\rm diff} = \mathcal{J}\frac{2\omega^2}{k^4}\left(\frac{n_{b0}}{n_{n0}n_{e0}}\right)^2\left(\sum_\alpha\frac{\partial\delta\mu}{\partial n_\alpha}n_{\alpha0}\right)^{-2}\sum_\alpha\frac{\mu_{\alpha0}n_{\alpha0}}{c^2}, \tag{12}$$

where $\mathcal{J} = J_{np}$ in normal matter and $\mathcal{J} = J_{en}$ in superconducting matter. In order to proceed to higher densities we generalize the results derived above to the case of $npe\mu$-matter. The ratio $\tau_{\rm diff}/\tau_\eta$ is plotted in Fig. 1 as a function of baryon number density $n_b$ for normal (dashed line) and strongly superconducting ($T \ll T_{\rm cp}$, solid line) matter.

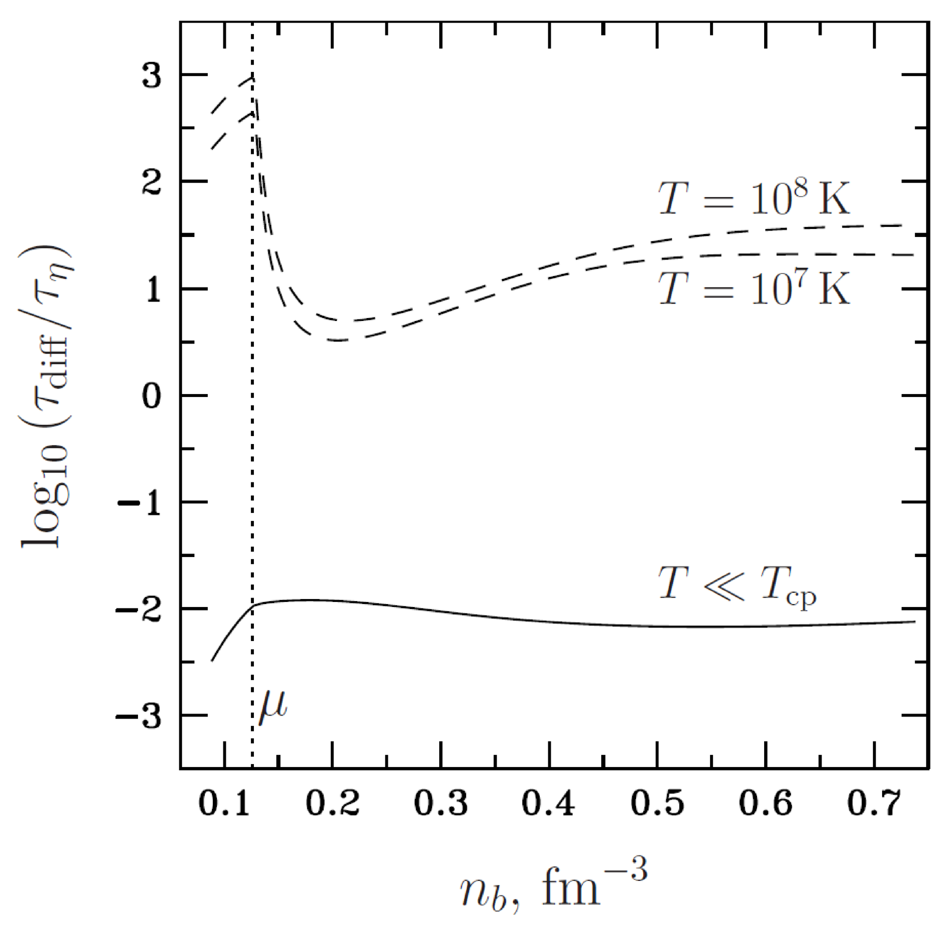


Figure 1: The ratio $\tau_{\rm diff}/\tau_\eta$ versus $n_b$ for sound waves in normal (dashed lines; temperature-dependent) and superconducting (solid line; almost temperature-independent) matter. Vertical dots denote the muon onset density.

In normal $npe$-matter particles are locked to each other: neutrons are locked to protons due to frequent collisions caused by strong interaction, while electrons are locked to protons by electromagnetic interaction. As a result, diffusion is inefficient. Appearance of muons allows charged particles to move with respect to each other, and the efficiency of diffusion increases strongly. However, our results imply that, anyway, diffusion is less effective in normal matter in the whole range of densities than the shear viscosity. At the same time, in superconducting matter neutrons are free to move with respect to protons and it is the diffusion, that becomes the dominant channel of energy losses.

## Results: $f$-, $p$-, and $g$-modes

The recently developed in [6, 10] formalism serves a relativistic generalization of the discussed above dissipative hydrodynamic equations accounting for particle diffusion, that is written in more convenient and easier-to-operate-with terms. Using this formalism, we calculate the damping times of relativistic $f$-, $p$-, and $g$-modes for an NS in the Cowling approximation (e.g., [13]). The oscillation eigenfunctions and eigenfrequencies in the absence of dissipation are calculated with our codes developed in [14]. Damping times versus eigenfrequency $\sigma \equiv \omega/(2\pi)$ for the first $(l = 2,\ m = 0)$ eigenmodes are shown in Fig. 2. Upper points represent dissipation due to shear viscosity, $\tau_\eta$, while lower points show diffusion damping times, $\tau_{\rm diff}$.

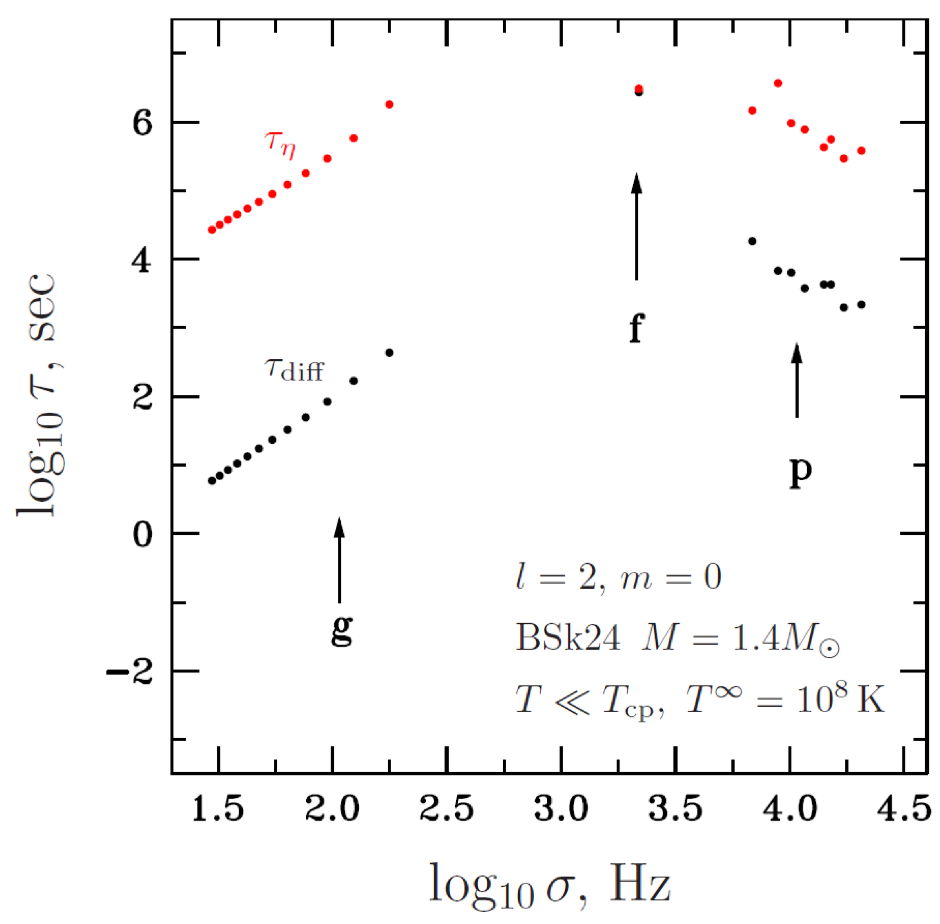


Figure 2: Damping times for the first $(l = 2,\ m = 0)$ $f$-, $p$-, and $g$-modes in superconducting NS due to shear viscosity (upper red points) and diffusion (lower black points) versus the mode eigenfrequency.

For $p$-modes $\tau_\eta$ exceeds $\tau_{\rm diff}$ by approximately two orders of magnitude, just like for sound waves. This is not surprising, since $p$-modes, being mainly restored by the pressure, are their close relatives. For $g$-modes, mainly restored by buoyancy, the difference between $\tau_\eta$ and $\tau_{\rm diff}$ is even larger — almost four orders of magnitude. Finally, the efficiency of diffusion for the $f$-mode is strongly suppressed, since this mode is almost incompressible and chemical potential imbalances are practically not perturbed in the course of oscillations.

## Results: $r$-modes

Rotating stars possess a special class of predominantly toroidal oscillations, restored mainly by the Coriolis force — $r$-modes. On one hand, these oscillations are unstable due to gravitational radiation [15, 16], but, on the other hand, they are damped by dissipative mechanisms, operating in the stellar matter. The instability window [17], that is the region on the $\nu - T^\infty$ plane ($\nu$ is the NS rotation frequency), where dissipation cannot counteract the $r$-mode growth, is populated by numerous sources [19] despite the theoretical predictions of the opposite [18]. Revealing some not yet identified strong dissipative mechanism could reconcile theory and observations. Here we examine, whether diffusion could serve as such a mechanism or not.

We calculate damping times for the most unstable $l = m = 2$ Newtonian $r$-mode in the Cowling approximation. In our calculations we assume that $\nu$ is small compared to the Kepler frequency, $\nu_{\rm K}$, and expand all the perturbations in the parameter $\nu/\nu_{\rm K}$. Fig. 3 shows the resulting instability curves (boundaries of the instability window), where the $r$-mode excitation is balanced by the shear viscosity only (dashed line) and by the combined action of shear viscosity and diffusion (thick solid line). Above the curves the $r$-mode is unstable.

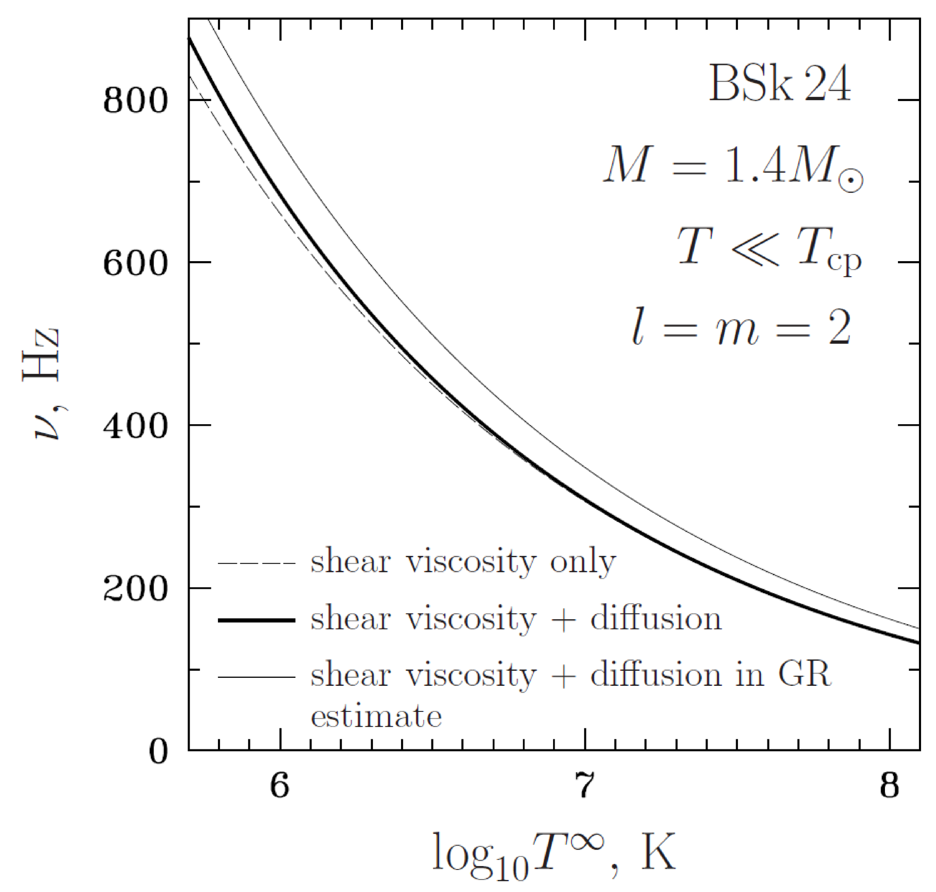


Figure 3: Instability curves for $l = m = 2$ $r$-mode in superconducting (but nonsuperfluid) NS. We extend the temperature range to extremely low values to illustrate the behaviour of the curves at large $\nu$.

Note that for the $r$-mode $\sigma \propto \nu$, while perturbations of thermodynamic quantities, in particular chemical potentials, are suppressed by a factor of $\nu^2/\nu_{\rm K}^2$. As a result, $1/\tau_{\rm diff}$ turns out to be $\propto \nu^2$ [see Eq. (9)], while $1/\tau_\eta$ does not depend on $\nu$. Thus, while at high $\nu$ diffusion is as effective as shear viscosity at lower $\nu$ diffusion is negligible. Consequently, in the Newtonian framework the effect of diffusion on the instability curve is smaller at higher $T^\infty$, since the curve $\nu(T^\infty)$ in Fig. 3 is a decreasing function of temperature.

Note, however, that in GR perturbations of chemical potentials are $\propto \nu/\nu_{\rm K}$ [20, 21], hence $1/\tau_{\rm diff}$ does not scale with $\nu$. To get an impression of how efficient diffusion in GR may be, we assumed that Newtonian approach and GR give similar results for $\tau_{\rm diff}$ at $\nu = \nu_{\rm K}$ (since chemical potential perturbations are not suppressed at $\nu = \nu_{\rm K}$). While the expansion parameter $\nu/\nu_{\rm K}$ is not small at $\nu = \nu_{\rm K}$, we can still formally solve the expanded oscillation equations at $\nu = \nu_{\rm K}$ and rescale the result to smaller $\nu$. Then we calculate $\tau_{\rm diff}$ at $\nu = \nu_{\rm K}$ in the Newtonian framework and assume that in GR $\tau_{\rm diff}$ equals this value at any $\nu$. The resulting GR instability curve due to diffusion and shear viscosity is shown by thin solid line and noticeably differs from the thick one. We emphasize that this is only an estimate, an accurate calculation will be published elsewhere.

## Conclusion

We propose that particle diffusion can be a very efficient dissipative mechanism in NSs. We compare damping of sound waves, as well as of $f$-, $p$-, $g$-, and $r$-modes due to diffusion and shear viscosity in NSs composed of neutrons, protons, and leptons. We find that, when protons are normal, the effect of diffusion on stellar oscillations is relatively small and can be ignored for all modes except for $g$-modes. In contrast, for superconducting protons diffusion leads to the very fast damping of oscillations, especially in the case of sound waves, $p$- and $g$- modes, leaving shear viscosity (which is believed to be the key dissipative mechanism) far behind. Our results imply that damping times of all oscillation modes should be revisited. Every physical phenomenon, dealing with the development of one or another hydrodynamic instability has to be reconsidered in order to check whether these instabilities can survive in the presence of the discussed here powerful diffusion dissipation. We therefore conclude that diffusion may have an important effect on the interpretation of gravitational signal produced in binary NS late inspirals, glitches, as well as on the interpretation of observational properties of rapidly rotating NSs.

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